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RELATION BETWEEN THE LAGRANGIAN AND EULERIAN DESCRIPTIONS OF TURBULENCE

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Let us consider a volume V filled with incompressible fluid. The volume can be either bounded or unbounded. Specifically, the fluid can fill the entire space. The boundaries can vary with time, but this variation must not depend on the motion of the fluid itself. This excludes the stream with a free surface and also the case of a vessel with elastic walls.

The position at the instant t of a fluid particle which initially occupied the position $\mathbf{a}$ will be denoted by $\xi(t, \mathbf{a})$. The condition of incompressibility is

$$\frac{D\xi}{D\mathbf{a}} = 1 \qquad (1)$$

The left side of this equation is a transformation Jacobian. The state of the fluid is characterized by the quantities $\sigma^{(k)}(t, \mathbf{a})$, $(k = 1, 2, \ldots)$, each of which can denote a set of fields (e.g. the velocity, vortex, pressure and impurity concentration fields, etc.). In Eulerian coordinates $\sigma^{(k)} = \sigma_E^{(k)}(t, \mathbf{x})$.

Let us isolate n fluid particles which occupied the positions $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}$ at the initial instant. The expression $P_n(t, \mathbf{x}^{(1)}, s^{(1)}, \ldots, \mathbf{x}^{(n)}, s^{(n)} \mid \mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}) =$

$$= \left\langle \prod_{k=1}^{n} \delta(\boldsymbol{\xi}(t, \mathbf{a}^{(k)}) - \mathbf{x}^{(k)}) \, \delta(\sigma^{(k)}(t, \mathbf{a}^{(k)}) - s^{(k)}) \right\rangle \quad (2)$$

($\delta(\mathbf{x})$ is a delta function)

defines the combined probability density of the values of $\boldsymbol{\xi}$ and $\sigma^{(k)}$ at the instant t; the angle brackets denote averaging over the ensemble of stream realizations (for a fixed law of boundary motion). In the Eulerian context

$$F_n(t, \mathbf{x}^{(1)}, s^{(1)}, \ldots, \mathbf{x}^{(n)}, s^{(n)}) = \left\langle \prod_{k=1}^{n} \delta(\sigma_E^{(k)}(t, \mathbf{x}^{(k)}) - s^{(k)}) \right\rangle \quad (3)$$

Here F_n is the probability density of the values of $\sigma^{(k)}$ at fixed points in space. For an incompressible fluid we have the basic formula

$$\int_V \cdots \int_V P_n \, d^3a^{(1)} \ldots d^3a^{(n)} = F_n \quad (4)$$

To prove this we substitute (2) into the left side of (4) and change the order of integration and averaging (this is possible in the case of a fixed law of boundary motion). We have

$$\int_V \cdots \int_V P_n \, d^3a^{(1)} \ldots d^3a^{(n)} = \left\langle \prod_{k=1}^{n} \int \delta(\boldsymbol{\xi}(t, \mathbf{a}) - \mathbf{x}^{(k)}) \, \delta(\sigma^{(k)}(t, \mathbf{a}) - s^{(k)}) \, d^3a \right\rangle \quad (5)$$

Let us convert from integration over $\mathbf{a}$ to integration over $\boldsymbol{\xi}$. By virtue of incompressibility condition (1), the transformation Jacobian is equal to unity. Since

$$\sigma^{(k)}(t, \mathbf{a}) = \sigma_E^{(k)}(t, \mathbf{x}) \quad \text{for} \quad \boldsymbol{\xi}(t, \mathbf{a}) = \mathbf{x} \quad (6)$$

it follows that the right sides of (5) and (3) coincide. Relation (4) has therefore been proved.

In [1] we used (without proof) the special case of formula (4) where the role of σ is played by the vortex field. Taking the velocity field as our σ, we find that for $n=1$ formula (4) yields

$$\int_V P_1(t, \mathbf{x}, \mathbf{v} \mid \mathbf{a}) \, d^3a = F_1(t, \mathbf{x}, \mathbf{v}) \quad (7)$$

Specifically, for a homogeneous stream

$$P_1(t, \mathbf{x}, \mathbf{v} \mid \mathbf{a}) = P_1(t, \mathbf{x} - \mathbf{a}, \mathbf{v} \mid 0) \quad (8)$$

and F_1 does not depend on $\mathbf{x}$. The density of the velocity distribution for a single fluid particle is

$$W_1(t, \mathbf{v}) = \int P_1(t, \mathbf{x}, \mathbf{v} \mid \mathbf{a}) \, d^3x \quad (9)$$

From (7)–(9) we obtain

$$W_1(t, \mathbf{v}) = F_1(t, \mathbf{v}) \quad (10)$$

Thus, in the case of a homogeneous stream of incompressible fluid the density of the velocity distribution of a specified fluid particle coincides with the density of the velocity distribution at a particular point. The same result was obtained in a different way in [2]. The above proof shows that (10) remains valid for any hydrodynamic field. Specifically, in estimating the parameters of the asymptotic form of the energy spectrum of a turbulent stream for large wave numbers in [3], we assumed, in fact, that the dispersions of the straining rate tensor in the Lagrangian and Eulerian descriptions were the same. We have now proved this fact.

A simple transformation of (4) yields the following expression for the relative motion of fluid particles in a homogeneous stream:

$$\int W_L(t, \mathbf{r}, \mathbf{u} \mid \mathbf{r}_0)\, d^3 r_0 = W_E(t, \mathbf{r}, \mathbf{u}) \tag{11}$$

Here W_L is the combined density of the distributions of the velocity difference and of the distance between fluid particles initially separated by the distance $\mathbf{r}_0$; W_E is the density of the velocity difference distribution at two fixed points the distance $\mathbf{r}$ apart. Formula (4) can also be used to obtain several other new relations.

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ON THE STATIC THEORY OF TWO-DIMENSIONAL TURBULENCE

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In contrast to three-dimensional motions, two-dimensional motions have not only the usual energy integral, but also an integral of motion which is quadratic in the velocity, namely the square of the curl of the velocity field. As is shown in [1, 2], this fact ensures the existence of a solution of the hydrodynamics equations with a normal (Gaussian) distribution of the velocity field probabilities with a spectrum different from white noise.

Our purpose in the present paper is to determine the characteristic of such a distribution, i. e. the correlation (structural) function of the fields under investigation, directly from the hydrodynamics equations.

Let us consider the two-dimensional motion of an incompressible inviscid turbulent fluid in the xy-plane. We assume that the turbulence is stationary in time and homogeneous and isotropic in space. The motion of the fluid is described by the stream function $\psi(\mathbf{r}, t)$ which satisfies the equation

$$\frac{\partial}{\partial t}\Delta\psi = \{\Delta\psi, \psi\}, \qquad \{\varphi, \psi\} = \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x} \tag{1}$$

Here $\{\varphi, \psi\}$ are the Poisson brackets and Δ is the two-dimensional Laplacian. The velocity field is defined by the vector $(-\partial\psi / \partial y,\ \partial\psi / \partial x)$.